High-order well-balanced methods for systems of balance laws: a control-based approach

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The main goal of this work is to develop high-order well-balanced schemes for 1d systems of balance law:

$$u_t + f(u)_x = S(u)\, H_x, \tag{1}$$

where $u(x,t)$ takes values in $\mathbb{R}^N$, $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the flux function, $S : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and H is a known function from $\mathbb{R}$ to $\mathbb{R}$.

A general methodology for developing such numerical methods on the basis of a standard reconstruction operator was introduced in [1] (see also [2]). The implementation of this strategy requires the computation, at every cell and at every time step, of the stationary solution whose cell average is equal to the numerical approximation already obtained. Since solving these problems can be difficult and/or expensive, we introduce some optimization techniques to rewrite them as control problems for the initial condition of an ODE system: given the approximation u_i^n at time t_n at the cell $I_i = [x_{i-1/2}, x_{i+1/2}]$, look for the solution of the control problem

$$\min_{u_0 \in \mathbb{R}^N} \left\| \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x)\, dx - u_i^n \right\|^2, \tag{2}$$

subject to:

$$\begin{cases} f(u)_x = S(u)\, \dfrac{dH}{dx}, \\ u(x_{i-\frac{1}{2}}) = u_0. \end{cases} \tag{3}$$

In order to solve numerically these control problems, the gradient descent method and a conjugate gradient-type method are considered. Different strategies for the step selection are also discussed. The introduced technique will be applied to different systems of balance laws including the shallow water model.

References

[1] Manuel J. Castro, José M. Gallardo, Juan A. López-García, and Carlos Parés. Well-balanced high order extensions of godunov's method for semilinear balance laws. *SIAM Journal on Numerical Analysis*, 46(2):1012–1039, 2008.

[2] Manuel J. Castro, Tomás Morales de Luna, and Carlos Parés. Well-balanced schemes and path-conservative numerical methods. In *Handbook of Numerical Analysis*, volume 18, pages 131–175. Elsevier, 2017.