# A Static Condensation Algorithm for Time-Implicit discretizations of Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods

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***Abstract.*** *We present a nodal (collocation) time-implicit Discontinuous Galerkin Spectral Element Method (DGSEM) with Gauss-Lobatto (GL) points, and show that it can be formulated as a Schur complement problem and solved using the static condensation method. By doing this, the linear system size is reduced, specially for high orders of accuracy, maintaining the advantageous properties of orthogonal basis expansions. Significant speed-ups are obtained as compared to previous explicit and implicit implimentations.*

*Traditionally, high-order Discontinuous Galerkin (DG) methods have been implemented using explicit time-integration methods. These time-integration schemes perform well when the spatial and temporal scales are similar, but are very inefficient for solving steady-state problems and very stiff equations because of time-step restrictions. Although important advances have been made[2,3], it is still a challenge to develop efficient time-implicit DG solvers able to overcome the severe time-step restriction of explicit schemes.*

*Static condensation is an efficient solution technique that was originally developed for solving linear systems that arise from time-implicit discretizations of the high-order Continuous Galerkin (CG) method [1]. However, in traditional DG methods it cannot be used directly. Certain techniques [2] have been developed to make modal DG methods suitable for static-condensation, but they imply the use of specially tailored basis functions with elementally non-orthogonal expansions.*

*In this work, we show that the system arising from the implicit time-integration of the GL-DGSEM can be reorganized in boundary (b) and interior (i) degrees of freedom as*

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{ib} \\ \mathbf{M}_{bi} & \mathbf{M}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \end{bmatrix}, \qquad (1)$$

*where* $\mathbf{M}_{ii}$ *is a block-diagonal matrix (Fig. 1.), while keeping elementally orthogonal expansions and their advantages.*

*This reorganization allows to formulate a Schur complement problem, where the solution can be obtained by solving a smaller condensed system of equations:*

$$\left[\mathbf{M}_{bb} - \mathbf{M}_{ib}\mathbf{M}_{ii}^{-1}\mathbf{M}_{bi}\right]\mathbf{q}_b = \mathbf{f}_b - \mathbf{M}_{ib}\mathbf{M}_{ii}^{-1}\mathbf{f}_i. \qquad (2)$$

The proposed method is tested for solving the compressible Euler and Navier-Stokes equations in several test cases, where we investigate the performance of direct and iterative (Krylov subspace) linear solvers for solving the global and condensed systems.

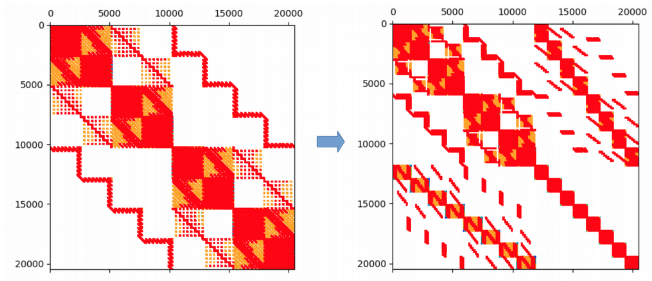

***Figure 1.*** *Original global matrix (a) and reorganized matrix (b) for a GL-DGSEM problem.*

We show that the ratio of the condensed system size to the global system size improves for high orders of accuracy and that it is problem dependent (Fig. 2). Furthermore, we provide a thorough study of the conditions where the static condensation algorithm is useful. In those conditions, significant speed-ups are obtained for high-order approxmations as compared to previous explicit and implicit implementations (Fig. 3).

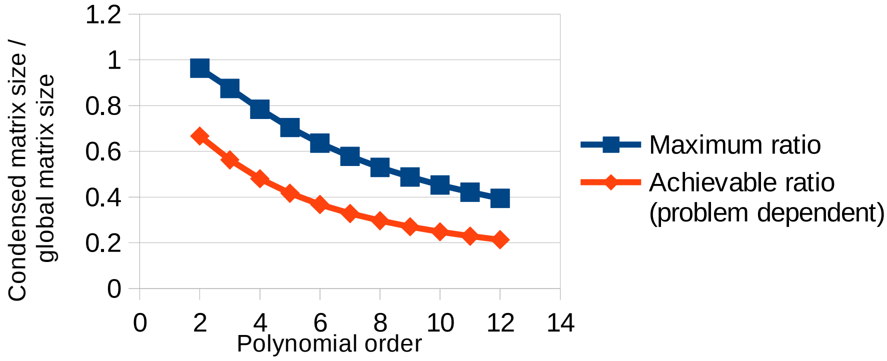


***Figure 2.*** *Ratio of the condensed system size to the global system size.*

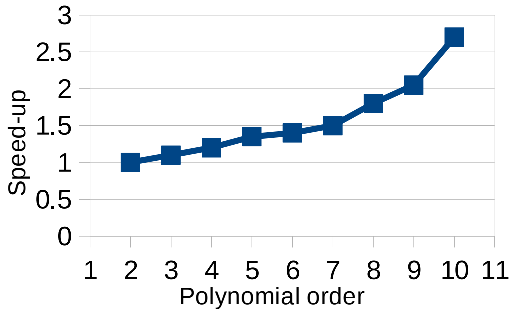


***Figure 3.*** *Speed-up in solving the linear system with a sparse LU method (MKL PARDISO). Global system time / Condensed system time.*

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