A free-energy stable nodal discontinuous Galerkin approximation with summation-by-parts property for the Cahn-Hilliard equation

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Abstract. *We present a nodal Discontinuous Galerkin (DG) spectral element method (DGSEM), which satisfies the summation-by-parts simultaneous-approximation-term (SBP-SAT) property, for the Cahn-Hilliard equation [2]. The SBP property permits us to show that the discrete free-energy is bounded and, as a result, the scheme is provably stable. The scheme and the stability proof are presented for general curvilinear three-dimensional hexahedral meshes and both exact and fully discrete integration in time. We use the Bassi-Rebay 1 (BR1) [3,4] scheme to compute interface fluxes, and an IMplicit-EXplicit (IMEX) scheme [1] to integrate in time for the fully discrete approximation.*

The Cahn-Hilliard equation in an arbitrary domain, Ω, with phase field ϕ, interfacial energy coefficient k, chemical potential $w = \psi'(\phi) - k\nabla^2\phi$ (with gradient $\vec{f} = \nabla w$) where $\psi(\phi)$ is the chemical free-energy, which is a polynomial function in ϕ, is

$$\phi_t = \nabla \cdot \left(M\vec{f}\right).$$

The free-energy,

$$\mathcal{F}(\phi,\nabla\phi) = \int_{\Omega} \left(\psi(\phi) + \frac{1}{2}k|\nabla\phi|^2\right) d\vec{x},$$

of the system satisfies a bound in time

$$\mathcal{F}(T) = \mathcal{F}(0) - \int_0^T \langle M\vec{f},\vec{f}\rangle dt \leq \mathcal{F}(0),$$

which can be viewed as a continuous statement of its stability.

We will show that the DGSEM approximation for the Cahn-Hilliard equation satisfies the discrete counterpart

$$\mathcal{F}^{T,N} \leq \mathcal{F}^{0,N} - \Delta t \sum_{E,n} \left\langle M\vec{F},\vec{F}\right\rangle_{E,N} \leq \mathcal{F}^{0,N},$$

where $\mathcal{F}^{i,N}$ and $\vec{F}$ are the approximations of $\mathcal{F}(T)$ and $\vec{f}$, respectively. In contrast to the continuous energy, the discrete statement is an inequality as a result of both the temporal and spatial dissipation introduced by the discretization.

We complement the analysis with two- and three-dimensional numerical experiments that assess the capabilities and convergence of the scheme. These experiments show that the approximation is spectrally accurate in space and design order accuracy in time. Fig. 1 shows an example of the time evolution of a mixture of two phases in three space dimensions using the code presented in [5] using the scheme analyzed here.

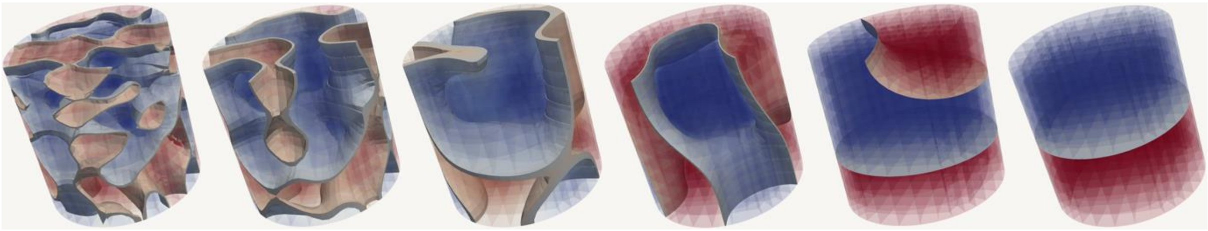

Figure 1 Evolution of the phases with time in a three-dimensional spinodal decomposition inside a cylinder.

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