Error estimation of linear numerical schemes on periodic meshes for transport equation

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Recently [1] proved a long-standing conjecture: for a DG solution of the initial problem for the transport equation $\partial u / \partial t + \partial u / \partial x = 0$ on the uniform mesh the numerical error $\varepsilon_h(t)$ has the estimate

$$\|\varepsilon_h(t)\| \le C_1 h^{k+1} + C_2 h^{2k+1} t \tag{1}$$

where k is the order of polynomials in use. Here $\varepsilon_h(t)$ is the difference between the numerical solution and the projection of the exact solution into the mesh subspace.

In the earlier paper [2] this fact was proved for $k = 1,2,3$ using the spectral analysis. To obtain a spectral representation one should find eigenvalues and eigenvectors of the matrix which depends on the wave number. The problem is that in general the solution can't be found analytically if the number of degrees of freedom (DOF) is greater than 4.

In [1] another approach was used – a special transform of the function into the mesh space. Let Π_h be an L_2-projection. In the sense of Π_h the DG scheme has approximation error of order k (if $k = 0$ the order is 1). Let L be the numerical derivative operator of the DG method. The authors of [1] constructed an operator $\tilde{\Pi}_h$ with the properties

1) $\|\tilde{\Pi}_h f - \Pi_h f\| \le O(h^{k+1})$, and
2) $\|L\tilde{\Pi}_h f - \tilde{\Pi}_h f'\| \le O(h^{2k+1})$, i.e. the approximation error in the sense of $\tilde{\Pi}_h$ is of order $2k+1$.

Then estimate (1) becomes a simple corollary of the triangle inequality and the stability of the scheme.

On non-uniform meshes the validity of estimate (1) (with substitution $h_{\max}$ for h) is a unique feature of the DG scheme. On the other hand, for uniform and periodic meshes (if the mesh period and structure do not change in refinement) DG is not the only scheme possessing the error estimate of the form

$$\|\varepsilon_h(t)\| \le C_1 h^{P} + C_2 h^{Q} t \tag{2}$$

for some P and Q. The aim of our research is to get optimal values of P and Q for a scheme given.

In order to establish (2) we can try to find $\tilde{\Pi}_h$ satisfying

$$\left\| L\tilde{\Pi}_h f - \tilde{\Pi}_h f' \right\| \le O(h^Q) \tag{3}$$

in the general form

$$\left(\tilde{\Pi}_h f\right)_{j,\xi} = \left(\Pi_h f\right)_{j,\xi} + \sum_{m=P}^{Q} h^m \mathfrak{C}_\xi^{(m)} \left(\Pi_h \left(\frac{d^m f}{dx^m}\right)\right)_{j,\xi} \tag{4}$$

where j is an index of the mesh block, ξ is an index of DOF and $\mathfrak{C}_\xi^{(m)}$ are the unknown coefficients. Note that the transform $\tilde{\Pi}_h$ used in [1] can be represented like this. As mentioned above, the existence of such $\tilde{\Pi}_h$ immediately implies (2).

Using the spectral analysis we show that the converse is also true: if the estimate is valid, there exists $\tilde{\Pi}_h$ of the form (4) that gives the Q^{th} order of approximation. So within this framework we can always get the optimal estimate of the numerical error. However in the multidimentional case there is no direct generalization of this fact. We have a 2D example of a scheme for which the numerical error satisfies estimate (2) but there is no operator $\tilde{\Pi}_h$ satisfying (3) of the form

$$\left(\tilde{\Pi}_h f\right)_{j,\xi} = \left(\Pi_h f\right)_{j,\xi} + \sum_{\substack{m_1,\ldots,m_d \ge 0 \\ P \le m_1+\ldots+m_d \le Q}} h^{m_1+\ldots+m_d} \mathfrak{C}_\xi^{(m_1,\ldots,m_d)} \left(\Pi_h \left(\frac{\partial^{m_1+\ldots+m_d} f}{\partial x_1^{m_1} \ldots \partial x_d^{m_d}}\right)\right)_{j,\xi}.$$

In the report we will present some cases of this analysis.

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References

1. Cao W., Zhang Z., Zou Q. Superconvergence of discontinuous Galerkin methods for linear hyperbolic equations // SIAM Journal on Numerical Analysis. 2014. Vol. 52, no. 5. P. 2555–2573.
2. Guo W., Zhong X., Qui C.-M. Superconvergence of discontinuous Galerkin and local discontinuous Galerkin methods: Eigen-structure analysis based on Fourier approach // Journal of Computational Physics. 2013. Vol. 235. P. 458–485.